Pricing Interest Rate Derivatives under Stochastic Volatility

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Abstract

This paper derives semi-closed-form solutions to a wide variety of interest rate derivatives prices under stochastic volatility in affine term structure models. We first derive the Frobenius series solution to the cross-moment generating function, and then invert the related characteristic function using the Gauss-Laguerre quadrature rule for the corresponding cumulative probabilities. This paper values options on discount bonds, coupon bond options, swaptions, interest rate caps, floors, and collars etc. The valuation approach suggested in this paper is found to be both accurate and fast and the approach compares favorably with some alternative methods in the literature. The valuation approach in this study can be used to price mortgage-backed securities (MBSs) and asset-backed securities (ABSs). The approach can also be used to value derivatives on other assets such as commodities. Finally, the approach in this paper is useful for the risk management of fixed income portfolios and financial planning in general. Future research could extend the approach adopted in this paper to some non-affine term structure models such as quadratic models etc.

Keywords: Characteristic functions, Frobenius series solution, Gauss-Laguerre quadrature rule, Interest rate derivatives, and Stochastic volatility

1. Introduction

It is widely accepted in the literature that interest rate volatility is stochastic (see, among others, Andersen and Lund (1997), Ball and Torous (1999), and Kalimipalli and Susmel (2004)). For example, Figures 1 and 2 plot the time series of the weekly 3-month U.S. T-bill rates as well as their first order differences over the time period of January 1954 to December 2006 (source of data: H.15 release at the Federal Reserve Board). Table 1 shows the summary statistics for both series. It is evident from both the figures and the descriptive statistics that the interest rates series are not Gaussian and that their volatility is changing through time. The empirical evidence thus suggests that it is important to model the stochastic volatility of interest rates.

[Figures 1 and 2 are about here.]

[Table 1 is about here.]

Modeling interest rate volatility is also critical from an asset pricing perspective. Almost all fixed-income securities contain embedded options, for example, callable bonds and putable bonds etc. Since prices of the embedded options (e.g. call and put options) depend on interest rate volatility, measuring the sensitivity of a security's value to interest rate volatility is therefore central to the pricing of fixed-income securities. Second, interest rate volatility is vital to the valuation of all kinds of interest rate derivatives. Finally, modeling interest rate volatility has a wide variety of applications in managing fixed-income portfolios, including portfolio hedging, portfolio indexing, and portfolio immunization, to name just a few. Indeed, the ability to effectively model interest rate volatility is fundamental to virtually all areas of fixed-income security and portfolio analysis. Not surprisingly, the academic literature on modeling interest rate volatility and pricing interest rate derivatives has been growing fast.

In this paper, we provide semi-closed-form solutions to the prices of various interest rate derivatives under stochastic interest rate volatility. Different from all the studies reviewed below, our approach is for a stochastic volatility interest rate model where the interest rate volatility is modeled by a separate process (the Fong and Vasicek (1991) model in particular)[1]. In our approach, we first derive the Frobenius series solution to the moment generating function of the zero-coupon bond price. Then, we invert the related characteristic function using the Gauss-Laguerre quadrature rule to recover the corresponding cumulative probabilities, which are used to compute the prices of different interest rate derivatives. Numerical examples show that our approach is easier to implement, fast, and accurate, and it compares favorably with some alternative approaches in the literature. However, it is important to note that our approach need not be restricted to the interest rate model considered in this paper. In fact, it can be readily generalized to other affine term structure models as well, including those models in which stochastic volatility is not explicitly specified.

A number of recent studies have also examined the pricing of options on coupon bonds. Jamshidian (1989), for instance, argues that an option on a coupon bond can be decomposed into a portfolio of options on discount bonds. Provided that there exists a closed-form solution to discount bond option price, we can solve for the value of a coupon bond option analytically. Jamshidian illustrates his approach in the context of the constant volatility Vasicek (1977) model, and his approach is valid for any single-factor interest rate model.

Wei (1997) develops an approximation for coupon bond options prices based on closed-form solutions for the corresponding discount bond options and a duration measure defined in his paper. He applies his approach to the constant volatility model in Vasicek (1977) and the model in Cox, Ingersoll, and Ross (1985, CIR hereafter) where volatility, though not a constant, is not modeled as a separate factor. Munk (1999) extends the approach in Wei (1997) and develops a method known as the *stochastic duration approach*. Munk applies his method to a number of multi-factor models including the deterministic volatility models in Ho and Lee (1986), Hull and White (1990), and Heath, Jarrow, and Morton (1992), and the model in Longstaff and Schwartz (1992). Singleton and Umantsev (2002) propose an approximation to the prices of European options on coupon bonds where the underlying short rate is an affine combination of the CIR-type square root processes. Stochastic volatility of interest rates is present in their framework only because in a CIR-type process volatility depends on the level of state variable (e.g. short rate) and is thus time-varying. However, in this framework volatility can not move independently of the state variable. None of the works just reviewed models interest rate volatility as a distinct process. In contrast, in the Fong and Vasicek (1991, FV hereafter) model considered in this paper, interest rate volatility is modeled as a separate process (see equations (1) and (2) in Section 2 below).

The contribution of this paper is two-fold. First, we extend the work of Fong and Vasicek (1991) and show how to value a wide range of interest rate derivatives within their stochastic volatility model. Second, we provide a viable alternative to some other approaches to the pricing of interest rate derivatives, for example, the Monte Carlo method proposed in Clewlow and Strickland (1997). Our paper also complements the recent work of Chacko and Das (2002), who suggest a general approach to valuing interest rate derivatives within the affine framework, in that we provide semi-closed-form solutions to interest rate derivatives prices under stochastic volatility and present a number of numerical examples to illustrate our approach.

The rest of the paper is organized as follows. Section 2 introduces the interest rate model used in this paper. Section 3 provides the solutions to options on zero-coupon bonds. Section 4 extends the approach to value coupon bond options and discusses the valuation of swaptions, interest rate caps, floors, and collars. Section 5 presents some numerical examples to assess the accuracy and efficiency of the proposed approach. Finally, Section 6 concludes the article. The computational details are contained in the Appendices.

2. The Fong and Vasicek (1991) interest rate model

Fong and Vasicek (1991) explicitly incorporate the stochastic volatility of interest rates as a separate factor (in addition to the short rate itself) and propose a two-factor model of interest rates. Their model is introduced as follows.

Under the physical (or actual) probability measure P, the instantaneous nominal riskless interest rate is denoted by r_t, and is assumed to follow the diffusion processes

$$dr_t = \alpha(\bar{r} - r_t)dt + \sqrt{v_t}\,dW_t, \tag{1}$$

$$dv_t = \gamma(\bar{v} - v_t)dt + \xi\sqrt{v_t}\,dZ_t, \tag{2}$$

where the two Brownian motions W_t and Z_t are correlated with a coefficient of ρ. In equation (2), v_t denotes the instantaneous variance of the risk free rate r_t and is modeled using a distinct stochastic process, separate from the process for r_t in (1). It follows that the volatility of interest rate r_t is also stochastic. In equations (1) and (2), α and γ are the speed of mean reversion of factors r_t and v_t, respectively; and $\bar{r}$ and $\bar{v}$ can be interpreted as the long-run mean of r_t and v_t, respectively. The specification in equations (1) and (2) belongs to the class of *exponential affine* (or simply *affine*) term structure of interest rate models[2].

We write the processes in equations (1) and (2) under the risk-neutral probability measure Q as

$$dr_t = \left(\alpha(\bar{r} - r_t) + \lambda v_t\right)dt + \sqrt{v_t}\, d\hat{W}_t, \tag{3}$$

$$dv_t = \left(\gamma\bar{v} - (\gamma + \xi\eta)v_t\right)dt + \xi\sqrt{v_t}\, d\hat{Z}_t, \tag{4}$$

where $\hat{W}_t$ and $\hat{Z}_t$ are two Brownian motions under Q that have a correlation coefficient of ρ, and λ and η are the risk premiums associated with interest rate risk and volatility risk, respectively.

In this paper we choose to use the above FV interest rate model for the following three reasons. First, it is well known that the dynamics of interest rates can not be adequately captured by a single factor. Instead, at least a two-factor model should be used (see e.g. Litterman and Scheinkman (1991)). Second, the FV model is intuitive. In particular, it captures the mean reversion exhibited by both the level and the volatility of interest rates. Furthermore, the process for the interest rate variance v_t in equation (2) suggests that although a quiet market (where volatility is low) might become highly volatile in time, it will not do so abruptly; in contrast, a very unstable market (where volatility is high) may cool down quite quickly or become even more volatile over a short time period, as we would observe in reality. Third and most importantly, the FV model uses a separate process to model interest rate volatility. As a result, in this model interest rate level and interest rate volatility become *imperfectly* correlated, which is borne out empirically in data. For example, Trolle and Schwartz (2008) in their study of interest rate caps, swaptions, and LIBOR/swap rates find that although innovations to interest rates and innovations to their volatility are correlated, the correlation is far from perfect (see their Footnote 3). The FV model can accommodate this feature of the data.

The FV model can generate various shapes for the yield curve, as shown in Figures 3 – 5. The FV model also allows for a closed-form solution to zero-coupon (or discount) bond price. But the original solution given in Fong and Vasicek (1991) involves complex algebra and is difficult to implement in practice. Selby and Strickland (1995) introduce an alternative method to compute the discount bond prices in the FV model. The Selby and Strickland method is based on the Frobenius series and is both accurate and fast. Finally, using Monte Carlo simulation Clewlow and Strickland (1997) show how to price various interest rate derivatives within the FV model. In this paper, we extend the Frobenius series method proposed in Selby and Strickland (1995) to price a variety of interest rate derivatives under stochastic volatility in an affine term structure model. Our approach is precise and much speedier than the Monte Carlo simulation method suggested in Clewlow and Strickland (1997). In addition, our approach can be generalized to other affine interest rate models as well.

[Figures 3, 4, and 5 are about here.]

3. Pricing of discount bond options

This section illustrates the pricing of call options on zero-coupon bonds[3][4]. Consider a call option with a maturity of T years written on a zero-coupon bond of a maturity of θ years and with a strike price of K. We use $P(T,\theta)$ and $Call(t,T,\theta,K)$ to denote the price of the underlying zero-coupon bond and the time t price of this call option, respectively. Under the risk-neutral probability measure Q, the call option price is given as

$$\begin{aligned} Call(t,T,\theta,K) &= E_t^Q\left(e^{-\int_t^T r_s ds} (P(T,\theta)-K)^+ \right) \\ &= E_t^Q\left(e^{-\int_t^T r_s ds} P(T,\theta)\times 1_{P(T,\theta)\geq K} \right) - KE_t^Q\left(e^{-\int_t^T r_s ds} \times 1_{P(T,\theta)\geq K} \right), \end{aligned} \tag{5}$$

where $E_t^Q(\cdot)$ denotes the expectation under the measure Q conditional on all the information available up to time t. Using the two forward measures Q^T and Q^θ that are equivalent to the measure Q, and defined by their Radon-Nikodym derivatives as

$$\frac{dQ^M}{dQ} \equiv \frac{e^{-\int_t^T r_s ds} P(T,M)}{P(t,M)}; \quad \text{for } M = T \text{ or } \theta, \tag{6}$$

we can rewrite equation (5) equivalently as[5]

$$Call(t,T,\theta,K) = P(t,\theta)E_t^{Q^\theta}\left(1_{\ln P(T,\theta)\geq \ln K}\right) - KP(t,T)E_t^{Q^T}\left(1_{\ln P(T,\theta)\geq \ln K}\right), \tag{7}$$

To compute the probability $E_t^{Q^M}\left(1_{\ln P(T,\theta)\geq \ln K}\right)$ for M = T or θ in (7), we have to first calculate its corresponding moment generating function

$$E_t^{Q^M}\left(e^{\varphi \ln P(T,\theta)}\right), \tag{8}$$

where φ is a constant. In Appendix 1, it is shown that the moment generating function in equation (8) is equal to

$$\frac{1}{P(t,M)} E_t^Q\left(e^{-\int_t^T r_s ds} \times e^{-(\varphi D(T,\theta)+D(T,M))r_T} \times e^{(\varphi F(T,\theta)+F(T,M))v_T} \times e^{(\varphi G(T,\theta)+G(T,M))} \right), \tag{9}$$

for M = T or θ. The functions $D(t,s)$, $F(t,s)$, and $G(t,s)$ in (9) above are the functions consisting of the discount bond price $P(t,s)$ in the FV model. We refer the interested reader to Selby and Strickland (1995) for the details about these three functions.

For notational simplicity, we rewrite equation (9) as

$$E_t^{Q^M}\left(e^{\varphi \ln P(T,\theta)}\right) = \frac{1}{P(t,M)} E_t^Q\left(e^{-\int_t^T r_s ds} \times e^{-A_0 r_T} \times e^{B_0 v_T} \times e^{C_0} \right), \tag{10}$$

with $A_0 \equiv \varphi D(T,\theta) + D(T,M)$, $B_0 \equiv \varphi F(T,\theta) + F(T,M)$, and $C_0 \equiv \varphi G(T,\theta) + G(T,M)$.

Once we solve the moment generating function $E_t^{Q^M}\left(e^{\varphi \ln P(T,\theta)}\right)$, we can recover the cumulative probability $Q^M\left(\ln P(T,\theta) \geq \ln K\right)$ by applying the Fourier inversion transform

$$Q^M\left(\ln P(T,\theta) \geq \ln K\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \mathrm{Re}\left\{ \frac{E_t^{Q^M}\left(e^{i\varphi \ln P(T,\theta)}\right)}{i\varphi} K^{-i\varphi} \right\} d\varphi \tag{11}$$

where $\text{Re}\{\cdot\}$ denotes the real part of a complex number. In equation (11) the integrand is well-defined and the integral is convergent. Finally, although the integral in (11) can not be solved in closed-form, numerical techniques can be used to approximate its value (see e.g. Heston (1993)). In this paper, we follow Sullivan (2000, 2001) and Tahani (2004) and use the Gauss-Laguerre quadrature rule to compute the value of the integral in equation (11).

Before computing the moment generating function $E_t^{Q^M}\left(e^{\varphi \ln P(T,\theta)}\right)$, it is helpful to first value the following "generalized" zero-coupon bond price[6]

$$P(t,T,\psi,\varphi,\omega) = E_t^Q\left(e^{-\psi\int_t^T r_s ds} \times e^{-\varphi r_T} \times e^{-\omega v_T}\right). \tag{12}$$

Denote $\tau \equiv T-t$, we can rewrite the above "generalized" bond price as $P(\tau,\psi,\varphi,\omega)$. We assume the following functional form

$$P(\tau,\psi,\varphi,\omega) = e^{-A(\tau)r+B(\tau)v+C(\tau)}. \tag{13}$$

We show in Appendix 2 that the system of the ordinary differential equations (ODEs) satisfied by the functions $A(\tau)$, $B(\tau)$, and $C(\tau)$ is the following

$$A' = -\alpha A + \psi, \quad A(0) = \varphi; \tag{14}$$

$$B' = \frac{1}{2}\xi^2 B^2 - ((\gamma+\xi\eta)+\rho\xi A)B - \lambda A + \frac{1}{2}A^2, \quad B(0) = -\omega; \tag{15}$$

$$C' = -\alpha\bar{r}A + \gamma\bar{v}B, \quad C(0) = 0. \tag{16}$$

In the above system of ODEs, A' denotes $\frac{\partial A}{\partial \tau}$, B' and C' are defined analogously; and $A(0)=\varphi$, $B(0)=-\omega$, and $C(0)=0$ are the initial conditions.

The ODE in (14), which is for function *A*, can be solved in simple closed-form as $A(\tau) = \frac{\psi}{\alpha} + \left(\varphi - \frac{\psi}{\alpha}\right)e^{-\alpha\tau}$, and function *C* can be found by direct integration once we know both functions *A* and *B*. The difficult part lies in solving the ODE for *B* in (15), which is a *Riccati* equation. A Riccati equation is one type of nonlinear first-order ODE. In the current case, although function *B* can be found in closed-form, the solution is fairly complicated and contains complex algebra (see Tahani (2004)).

To overcome this difficulty, we follow Selby and Strickland (1995) and make a simple substitution

$$U(\tau) = \exp\left(-\frac{1}{2}\xi^2\int_0^\tau B(s)ds\right), \quad U(0) = 1, \quad U'(0) = \frac{\xi^2}{2}\omega. \qquad (17)$$

This substitution transforms the nonlinear ODE for *B* in (15) into an equivalent linear second-order ODE for *U*. Under this substitution, functions *B* and *C* can be rewritten as

$$B(\tau) = -\frac{2}{\xi^2}\frac{U'(\tau)}{U(\tau)}, \qquad (18)$$

$$C(\tau) = -\alpha\bar{r}\int_0^\tau A(s)ds - \frac{2\gamma\bar{v}}{\xi^2}\ln U(\tau). \qquad (19)$$

Therefore the solution to the “generalized” zero-coupon bond pricing formula in (13) amounts to evaluating $U(\tau)$ and $U'(\tau)$. A further substitution

$$U(\tau) = z^\beta S(z), \quad \tau = -\frac{1}{\alpha}\ln(z), \quad 0 \le z \le 1; \qquad (20)$$

where β is a constant to be determined, reduces the ODE for *U* to a *homogeneous* linear ODE of second order for *S*, which can be solved by using a Frobenius series solution method. Once we obtain the solution to *S*, we can retrace, substituting *S* back into equation (20) for *U*, and then substituting *U* back into (18) and (19) for functions *B* and *C*, respectively. For computational details, please refer to Appendix 3.

4. Valuation of options on coupon bonds and other derivatives

We now extend the valuation approach in the previous section to the pricing of coupon bond options and other interest rate derivatives.

4.1 Coupon bond options

Following Munk (1999), we approximate the price of a coupon bond option using an option on a zero-coupon bond that has the same *stochastic duration* as the coupon bond. In particular, consider a coupon bond paying a_i at time t_i, *i* = 1, 2, …, *n*. The price of this bond at any time $t < t_i$ is given by $H(t) = \sum_{i=1}^{n} a_i P(t, t_i),$ where $P(t, t_i)$ is the time t price of the discount bond that matures at time t_i. A straightforward application of Itô's lemma yields

$$\begin{aligned} \frac{dH(t)}{H(t)} &= \sum_{i=1}^{n} \bar{a}_i(t,t_i)\mu(t,t_i,r_t,v_t)dt + \xi\sqrt{v_t}\sum_{i=1}^{n}\bar{a}_i(t,t_i)F(t,t_i)dZ_t \\ &\quad - \sqrt{v_t}\sum_{i=1}^{n}\bar{a}_i(t,t_i)D(t,t_i)dW_t, \end{aligned} \tag{21}$$

where the weight $\bar{a}_i(t,t_i) = \dfrac{a_i P(t,t_i)}{H(t)}$, $\mu(t,t_i,r_t,v_t)$ is a function of the model's parameters, and the functions $D(t,t_i)$ and $F(t,t_i)$ are the functions involved in computing the zero-coupon bond price $P(t,t_i)$ in the FV model. Another direct application of Itô's lemma, for the zero-coupon bond price, results in

$$\frac{dP(t,s)}{P(t,s)} = \mu(t,s,r_t,v_t)dt + \xi\sqrt{v_t}F(t,s)dZ_t - \sqrt{v_t}D(t,s)dW_t. \tag{22}$$

The stochastic duration $\delta(t)$ is defined as the time to maturity of the discount bond that has the same *relative volatility* (please refer to Munk (1999) for a definition of relative volatility) as the coupon bond. More specifically, $\delta(t)$ is the solution to the following equation

$$\begin{aligned} &\left[\xi F(t,\delta(t)) - \rho D(t,\delta(t))\right]^2 + \left(1-\rho^2\right)D(t,\delta(t))^2 \\ &= \left[\sum_{i=1}^{n}\bar{a}_i(t,t_i)\left[\xi F(t,t_i) - \rho D(t,t_i)\right]\right]^2 + \left(1-\rho^2\right)\left[\sum_{i=1}^{n}\bar{a}_i(t,t_i)\,D(t,t_i)\right]^2. \end{aligned} \tag{23}$$

Clearly, equation (23) has to be solved numerically. Once $\delta(t)$ is computed, the price of the coupon bond option can be approximated by a multiple of the price of the option on a zero-coupon bond with a time to maturity equal to $\delta(t)$. More formally, the price of a call option on the coupon bond defined above is approximated by[7]

$$Call_{Coupon\ Bond}(t,T,K) \cong \zeta Call(t,T,t+\delta(t),\frac{K}{\zeta}), \tag{24}$$

where $\zeta = \frac{H(t)}{P(t, t+\delta(t))}$ and $Call(t, T, t+\delta(t), \frac{K}{\zeta})$ is the time t price of a call option with a maturity of T years and a strike price of $\frac{K}{\zeta}$ written on a discount bond of a maturity of

$t+\delta(t)$ years. The put option price can be approximated similarly.

4.2 Swaptions

A *swaption* (or option on an interest rate swap) is an option to exchange periodical fixed-rate payments for floating-rate payments. When the swap is created, the floating-rate payments have a present value equal to the notional principal of the swap. The swaption can therefore be valued as a coupon bond option with a strike price equal to the principal amount of the underlying swap (see Hull (2006)).

4.3 Interest rate caps, floors, and collars

An *interest rate cap* or *floor* provides the right to get payoffs at periodic dates called the *reset dates*. At each reset date, the interest rate cap/floor has a payoff that is the same as the payoff from a zero-coupon bond put/call option. Therefore, the interest rate cap/floor can be seen as a sequence of many puts/calls called *caplets* or *floorlets*, respectively. The cap/floor premium is then equal to the sum of the corresponding caplets/floorlets premiums. In particular, the price of an interest rate cap with reset dates $(\theta_i)_{1\le i\le n}$ and a cap rate r_{cap} is given by

$$Cap(t, (\theta_i)_{1\le i\le n}, r_{cap}) = (1 + r_{cap}\,\Delta\theta)\sum_{i=1}^{n-1} Put(t, \theta_i, \theta_{i+1,} K_{cap}), \qquad (25)$$

where $K_{cap} = \frac{1}{1 + r_{cap}\,\Delta\theta}$, $\Delta\theta$ is the reset interval, and $Put(t, \theta_i, \theta_{i+1,} K_{cap})$ is the price of a zero-coupon bond put option.

Similarly, the price of an interest rate floor with reset dates $(\theta_i)_{1\le i\le n}$ and a floor rate r_{floor} is given by

$$Floor(t,(\theta_i)_{1\le i\le n}, r_{floor}) = (1+r_{floor}\ \Delta\theta)\sum_{i=1}^{n-1} Call(t,\theta_i,\theta_{i+1,}K_{floor}), \qquad (26)$$

where $K_{floor} = \dfrac{1}{1+r_{floor}\ \Delta\theta}$ and $Call(t,\theta_i,\theta_{i+1,}K_{floor})$ is the price of a zero-coupon bond call option.

A *collar* is another popular interest rate derivative. It is simply a combination of a long position on an interest rate cap and a short position on an interest rate floor with the same characteristics (i.e., the same settlement dates and the same reset intervals etc.). It follows that the collar can simply be priced as the difference between the price of the long cap with a strike price K_{cap} and the price of the short floor with a strike price K_{floor}.

5. Numerical examples

This section presents some numerical examples to assess the accuracy and the efficiency of the proposed approximation with different sets of parameters. The benchmark prices are either given by closed-form solutions such as the Jamshidian (1989) formula in the case of constant volatility Vasicek (1977) model, or by a Monte Carlo simulation based on 100,000 paths in the case of the FV (1991) stochastic volatility model. The parameters values used in our experiment are close to those adopted in Clewlow and Strickland (1997), and they are also generally consistent with the historical behavior of interest rates.

First, we assess the approximation within the Vasicek (1977) constant volatility model. For clarity and completeness, we present below the interest rate process in the Vasicek (1977) model, which can be interpreted similarly to the FV model in equations (1) and (2)

$$dr_t = \alpha(\bar{r} - r_t)dt + vdW_t. \qquad (27)$$

We consider the following set of parameters: $\alpha = 1.2$, $\bar{r} = 0.095$, $r = 0.08$, and $v = 0.015$. We price a one-year at-the-money forward call on a five-year zero-coupon bond with a face value of \$1. The strike price K of this call option is \$0.6392. Table 2 shows how the approximate price converges to the benchmark price given by the Jamshidian (1989) formula for different quadrature orders as well as the associated absolute and relative errors. Note that an order of (as low as) 15 is sufficient to obtain a very accurate price.

[Table 2 is about here.]

The second example is a one-year at-the-money forward call on a two-year zero-coupon bond in the FV stochastic volatility model with the following set of parameters: $\alpha = 2$, $\bar{r} = 0.07$, $r = 0.08$, $\bar{v} = \mathrm{v} = 0.02$, $\gamma = 2$, $\lambda = 0.2$, $\xi = 0.0001$, $\eta = 0.1$, and $\rho = 0.2$. The strike price K of this call option is \$0.9322. The third example is a one-year at-the-money forward call on a five-year zero-coupon bond in the FV model with the following set of parameters: $\alpha = 2$, $\bar{r} = 0.095$, $r = 0.08$, $\bar{v} = \mathrm{v} = 0.015$, $\gamma = 2$, $\lambda = 0.2$, $\xi = 0.0001$, $\eta = 0.1$, and $\rho = 0.6$. The strike price K of this call option is \$0.6236. Tables 3 and 4 provide the approximate prices, the Monte Carlo prices, and their standard deviations for these two examples, respectively. It is shown in the tables that a high degree of accuracy can be achieved for a quadrature order of about 25.

[Tables 3 and 4 are about here.]

We now provide some results for options on coupon bonds. Consider a one-year call on a five-year coupon bond with a semiannual coupon of \$0.04 and a face value of \$1. Tables 5 and 6 use the Vasicek (1977) model since we have a closed-form solution provided by Jamshidian (1989) to which we can compare our approximation. In both tables, we use the same parameters values as those in Table 2. Table 5 shows the results for an at-the-money call with a strike price K = \$0.8767. The stochastic duration of this coupon bond is 3.5324 and the call price given by our approximation is \$0.0733076, while in comparison the call price given by the Jamshidian solution is \$0.0733027. Table 6 does a similar analysis for an in-the-money call with a strike price K = \$0.7970. The call price according to our approach is \$0.144771, while the Jamshidian price is \$0.144770.

[Tables 5 and 6 are about here.]

Within the FV (1991) stochastic volatility model, Tables 7 and 8 compare our approximation to a Monte Carlo simulation using the same parameters values as those in

Table 4. The stochastic duration of the coupon bond is 2.8825. In Table 7, the at-the-money strike price is $0.8557, the approximate call price is $0.0726108, while the Monte Carlo price is $0.0726402 with a standard deviation of 8.6275e-5. In Table 8, the in-the-money strike price is $0.8150, the approximate price is $0.109902, while the Monte Carlo price is $0.109801 with a standard deviation of 8.8149e-5.

[Tables 7 and 8 are about here.]

In all of the above numerical examples, the approximations based on the Frobenius series and the stochastic duration are found to be very accurate relative to either the closed-form solution given in Jamshidian (1989) or the Monte Carlo simulation. Finally, Table 9 contains a comparison of the computation time. The approximation method is found to be very fast, especially when compared to the Monte Carlo valuation method. Our approach can achieve a high degree of accuracy in less than one second of time on a standard computer.

[Table 9 is about here.]

6. Conclusion

This paper derives semi-closed-form pricing formulas for different interest rate derivatives under stochastic volatility. It extends the Fong and Vasicek (1991) valuation formula as well as the Selby and Strickland (1995) methodology. The main contribution consists of deriving the moment generating function of the zero-coupon bond as a Frobenius series. This allows us to easily price options on zero-coupon bonds by inverting the corresponding characteristic function using the Gauss-Laguerre quadrature rule. The approach is then applied to the pricing of coupon bond options, swaptions, interest rate caps, floors, and collars etc. The numerical analysis conducted shows that the combination of the Frobenius series and the quadrature rules provides a very accurate and fast valuation of a wide variety of interest rate derivatives. Although the approximation is developed for the FV model, it can be readily adapted to any other affine term structure models, including those models in which stochastic volatility of interest rates is not explicitly specified, provided that we adjust the "generalized" zero-coupon bond price defined in equation (12) accordingly.

Our approach has many applications. To name just a few, first, our approximation method can be useful for pricing complex interest rate derivatives, such as mortgage-backed securities (MBSs) and asset-backed securities (ABSs) since interest rate volatility is a key ingredient in determining the value of the prepayment option embedded in these securities. Second, our approach can be extended to value derivatives on other assets such as commodities. Third, our method can be applied to the risk management of fixed-income portfolios and financial planning in general. Extending our approach to these applications is an interesting venue for our future research.

Notes

1. Section 2 reports some empirical evidence in support of modeling the interest rate volatility as a separate process, different from the stochastic process that models the interest rate level.

2. Dai and Singleton (2000) conduct a thorough specification analysis of various affine interest rate models. Also see Duffie and Kan (1996).

3. Put options on zero-coupon bonds can be valued using the put-call parity formula: $Put(t,T,\theta,K) = Call(t,T,\theta,K) + KP(t,T) - P(t,\theta)$, where $Put(t,T,\theta,K)$ denotes the time t price of a put option with a maturity of T years and a strike price of K written on a zero-coupon bond of a maturity of θ years.

4. To save space, only call options are considered throughout the paper. The prices of put options on either discount bonds or coupon bonds can be easily calculated by applying the put-call parity (as shown in Endnote 3 above). We have also conducted the numerical analysis in Section 5 using puts. These results (not reported but are available upon request) support the findings in the paper.

5. See Geman, El Karoui, and Rochet (1995) for the derivation of the forward measure and its use in option pricing.

6. This is the cross-moment generating function of the joint distribution of the short rate, its volatility and its time-integral. In defining and computing this function, we actually extend the method in Selby and Strickland (1995) since they only compute zero-coupon bond prices, yet we calculate the moment generating function of zero-coupon bonds, which is used subsequently to compute interest rate derivatives prices.

7. The approach of Jamshidian (1989) in which an option on a coupon bond is decomposed into a portfolio of options on discount bonds with adjusted strike prices is applicable only to one-factor models, so we can not use his approach here since the FV model is a two-factor model.

Appendices: Technical details

Appendix 1: Derivation of the moment generating function

The moment generating function of $\ln P(T,\theta)$ under the forward measure Q^M for $M = T$ or θ is given by

$$
\begin{aligned}
&E_t^{Q^M}\left(e^{\varphi \ln((T,\theta)}\right) = E_t^{Q^M}\left(e^{-\varphi D(T,\theta) r_T + \varphi F(T,\theta) v_T + \varphi G(T,\theta)}\right) \\
&= \frac{1}{P(t,M)} E_t^Q\left(e^{-\int_t^T r_s ds} \times P(T,M) \times e^{-\varphi D(T,\theta) r_T + \varphi F(T,\theta) v_T + \varphi G(T,\theta)} \right) \\
&= \frac{1}{P(t,M)} E_t^Q\left(e^{-\int_t^T r_s ds} \times e^{-(\varphi D(T,\theta) + D(T,M)) r_T} \times e^{(\varphi F(T,\theta) + F(T,M)) v_T} \times e^{(\varphi G(T,\theta) + G(T,M))} \right) \qquad \text{(A.1)} \\
&= \frac{1}{P(t,M)} E_t^Q\left(e^{-\int_t^T r_s ds} \times e^{-A_0 r_T} \times e^{B_0 v_T} \times e^{C_0} \right),
\end{aligned}
$$

where functions A_0, B_0 and C_0 are defined in equation (10).

Appendix 2: Derivation of the ODEs in equations (14) – (16)

Using the Feynman-Kac theorem, the cross-moment generating function defined in equation (12) must solve the following PDE

$$
\begin{aligned}
&\frac{1}{2} v \frac{\partial^2 P}{\partial r^2} + \frac{1}{2}\xi^2 v \frac{\partial^2 P}{\partial v^2} + \rho \xi v \frac{\partial^2 P}{\partial v \partial r} + (\alpha \bar{r} - \alpha r + \lambda v)\frac{\partial P}{\partial r} \\
&\qquad + (\gamma \bar{v} - (\gamma + \xi \eta) v)\frac{\partial P}{\partial v} - \psi r P = \frac{\partial P}{\partial \tau}, \qquad \text{(A.2)}
\end{aligned}
$$

with the boundary condition $P(\tau = 0, \psi, \varphi, \omega) = e^{-\varphi r - \omega v}$. Assuming the following functional form in (13), $P(\tau, \psi, \varphi, \omega) = e^{-A(\tau) r + B(\tau) v + C(\tau)}$, taking derivatives and substituting the resulted derivatives into the above PDE, we can easily show that functions *A*, *B* and *C* must solve the ODEs in equations (14) – (16).

Appendix 3: Derivation of the Frobenius series solution

Define $U(\tau)=\exp\left(-\frac{1}{2}\xi^2\int_0^\tau B(s)ds\right)$. Substituting the definition of $U(\tau)$ into the ODE in equation (15), it is easy to see that U must solve the following second-order linear ODE

$$U''+(\rho\xi A+\xi\eta+\gamma)U'-\frac{\xi^2}{2}(\lambda A-\frac{1}{2}A^2)U=0, \tag{A.3}$$

with $U(0)=1$ and $U'(0)=\frac{\xi^2}{2}\omega$. Now consider the function $S(z)=z^{-\beta}U(\tau)$ where $z=e^{-\alpha\tau}$, it can be shown that S solves the following homogeneous second-order ODE

$$zS''+(\overline{\alpha}+\overline{\beta}z)S'+(\overline{\gamma}+\overline{\delta}z)S=0, \tag{A.4}$$

with $S(1)=1$ and $S'(1)=-\beta-\frac{\xi^2}{2\alpha}\omega$, and where

$$\begin{aligned}
\overline{\alpha}&=2\beta-\kappa+1,\\
\overline{\beta}&=\frac{\rho\xi}{\alpha^2}(\psi-\alpha\varphi),\\
\overline{\delta}&=\frac{\xi^2}{4\alpha^4}(\psi-\alpha\varphi)^2,\\
\overline{\gamma}&=\left(\frac{\rho\xi\beta}{\alpha^2}-\frac{\xi^2}{\alpha^4}(\psi-\alpha\lambda)\right)(\psi-\alpha\varphi),\\
\kappa&=\frac{\xi\eta+\gamma}{\alpha}+\frac{\rho\xi}{\alpha^2}\psi,\\
\kappa_1&=\frac{\xi^2\psi}{2\alpha^3}\left(\frac{\psi}{2\alpha}-\lambda\right),\\
\beta&=\frac{\kappa}{2}-\frac{1}{2}\sqrt{\kappa^2-4\kappa_1}.
\end{aligned} \tag{A.5}$$

Assume the following Frobenius series $S(z;\varepsilon)=z^{\varepsilon}\sum_{n=0}^{+\infty}q_n z^n$ as a solution to the homogeneous ODE in equation (A.4) above. It can be shown that $\varepsilon=0$ or $\varepsilon=1-\overline{\alpha}$. Moreover, the coefficients q_n are given recursively by

$$
\begin{aligned}
q_1 &= \frac{-(\bar{\gamma} + \bar{\beta}\varepsilon)\, q_0}{(1+\varepsilon)(\varepsilon + \bar{\alpha})}, \\
q_n &= \frac{-\bar{\delta}\, q_{n-2} - \left(\bar{\gamma} + \bar{\beta}(n-1+\varepsilon)\right) q_{n-1}}{(n+\varepsilon)(n-1+\varepsilon+\bar{\alpha})} \quad ; \quad \text{for } n \geq 2,
\end{aligned} \tag{A.6}
$$

where q_0 is any arbitrary constant. The general solution to the function S can then be written as $S(z) = aS(z;0) + bS(z;1-\bar{\alpha})$ where

$$
\begin{aligned}
b &= \frac{S'(1;0) - S'(1)S(1;0)}{S(1;1-\bar{\alpha})S'(1;0) - S'(1;1-\bar{\alpha})S(1;0)}, \\
a &= \frac{1 - bS(1;1-\bar{\alpha})}{S(1;0)}.
\end{aligned} \tag{A.7}
$$

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Table 1: Summary statistics of the interest rate series

	Min	1Q	Median	3Q	Max
3-Month US T-Bill	0.58	3.15	4.94	6.45	16.76
First order change	-1.82	-0.05	0.00	0.06	1.92

	Mean	Std. Dev.	Skewness	Kurtosis
3-Month US T-Bill	5.175	2.808	1.104	4.828
First order change	0.001	0.200	-0.681	25.761

The interest rates data are weekly and are taken from the H.15 release at the Federal Reserve Board. The sample period is from January 1954 to December 2006.

Table 2: Zero-coupon call option in the Vasicek (1977) model

Order	Price	Absolute error	Relative error
10	1.434E-02	3.36E-04	2.29E-02
11	1.450E-02	1.68E-04	1.15E-02
12	1.459E-02	7.91E-05	5.39E-03
13	1.464E-02	3.47E-05	2.36E-03
14	1.466E-02	1.41E-05	9.63E-04
15	1.467E-02	5.35E-06	3.65E-04
16	1.467E-02	1.85E-06	1.26E-04
17	1.467E-02	5.65E-07	3.85E-05
18	1.467E-02	1.90E-07	1.29E-05
19	1.467E-02	3.98E-08	2.71E-06
20	1.467E-02	1.67E-10	1.14E-08
21	1.467E-02	1.55E-09	1.06E-07
22	1.467E-02	8.44E-09	5.75E-07
23	1.467E-02	5.58E-09	3.80E-07
24	1.467E-02	8.19E-09	5.58E-07
25	1.467E-02	1.65E-08	1.12E-06
26	1.467E-02	1.12E-10	7.61E-09
27	1.467E-02	1.05E-07	7.17E-06
28	1.467E-02	6.63E-08	4.52E-06
29	1.467E-02	5.40E-07	3.68E-05
30	1.467E-02	5.06E-07	3.45E-05
Benchmark price	1.467E-02		

In the table, *Order* is the order of the Gauss-Laguerre quadrature rule used in our approximation, *Benchmark price* is the call price given by the closed-form pricing formula in Jamshidian (1989), *Price* refers to the call price obtained by our semi-closed-form solution, *Absolute error* is the absolute value of the difference between the Benchmark price and Price, and *Relative error* is calculated as $\frac{|\text{Benchmark price} - \text{Price}|}{\text{Benchmark price}}$.

Table 3: Zero-coupon call option in the FV (1991) model

Order	Price	Absolute error	Relative error
10	7.497E-03	3.00E-03	2.85E-01
11	8.006E-03	2.49E-03	2.37E-01
12	8.452E-03	2.04E-03	1.94E-01
13	8.836E-03	1.66E-03	1.58E-01
14	9.163E-03	1.33E-03	1.27E-01
15	9.436E-03	1.06E-03	1.01E-01
16	9.662E-03	8.31E-04	7.92E-02
17	9.845E-03	6.47E-04	6.17E-02
18	9.992E-03	5.00E-04	4.76E-02
19	1.011E-02	3.84E-04	3.66E-02
20	1.020E-02	2.93E-04	2.80E-02
21	1.027E-02	2.24E-04	2.14E-02
22	1.032E-02	1.72E-04	1.64E-02
23	1.036E-02	1.33E-04	1.26E-02
24	1.039E-02	1.04E-04	9.92E-03
25	1.041E-02	8.35E-05	7.96E-03
26	1.042E-02	6.88E-05	6.56E-03
27	1.043E-02	5.84E-05	5.57E-03
28	1.044E-02	5.15E-05	4.91E-03
29	1.045E-02	4.59E-05	4.37E-03
30	1.045E-02	4.35E-05	4.14E-03
MC price	1.049E-02		
Std. Dev.	5.111E-05		

In the table, *MC price* is the call price given by a Monte Carlo simulation based on 100,000 paths for the FV (1991) stochastic volatility model, and *Std. Dev.* is the associated standard deviation. The other terms used in the table are defined in Table 2.

Table 4: Zero-coupon call option in the FV (1991) model

Order	Price	Absolute error	Relative error
10	4.958E-03	1.97E-03	2.85E-01
11	5.294E-03	1.64E-03	2.36E-01
12	5.588E-03	1.34E-03	1.94E-01
13	5.841E-03	1.09E-03	1.57E-01
14	6.057E-03	8.73E-04	1.26E-01
15	6.237E-03	6.93E-04	1.00E-01
16	6.386E-03	5.44E-04	7.85E-02
17	6.506E-03	4.23E-04	6.11E-02
18	6.603E-03	3.27E-04	4.71E-02
19	6.679E-03	2.50E-04	3.61E-02
20	6.739E-03	1.91E-04	2.75E-02
21	6.784E-03	1.45E-04	2.10E-02
22	6.819E-03	1.11E-04	1.60E-02
23	6.844E-03	8.55E-05	1.23E-02
24	6.863E-03	6.68E-05	9.64E-03
25	6.876E-03	5.33E-05	7.69E-03
26	6.886E-03	4.37E-05	6.31E-03
27	6.893E-03	3.69E-05	5.33E-03
28	6.897E-03	3.24E-05	4.68E-03
29	6.901E-03	2.88E-05	4.15E-03
30	6.902E-03	2.72E-05	3.93E-03
MC price	6.930E-03		
Std. Dev.	3.351E-05		

The terms used in the table are defined in Table 3.

Table 5: Coupon bond call option in the Vasicek (1977) model

Order	Price	Absolute error	Relative error
20	7.330784E-02	5.15E-06	7.03E-05
21	7.330785E-02	5.17E-06	7.05E-05
22	7.330789E-02	5.20E-06	7.10E-05
23	7.330790E-02	5.21E-06	7.11E-05
24	7.330791E-02	5.22E-06	7.12E-05
25	7.330786E-02	5.18E-06	7.06E-05
26	7.330787E-02	5.19E-06	7.08E-05
27	7.330794E-02	5.25E-06	7.16E-05
28	7.330784E-02	5.16E-06	7.03E-05
29	7.330778E-02	5.09E-06	6.95E-05
30	7.330759E-02	4.90E-06	6.69E-05
Benchmark price	7.330269E-02		

The terms used in the table are defined in Table 2.

Table 6: Coupon bond call option in the Vasicek (1977) model

Order	Price	Absolute error	Relative error
20	1.447702E-01	5.06E-07	3.49E-06
21	1.447701E-01	3.76E-07	2.60E-06
22	1.447700E-01	3.40E-07	2.35E-06
23	1.447700E-01	3.55E-07	2.45E-06
24	1.447701E-01	3.70E-07	2.56E-06
25	1.447700E-01	3.35E-07	2.31E-06
26	1.447700E-01	3.31E-07	2.29E-06
27	1.447699E-01	1.63E-07	1.13E-06
28	1.447702E-01	4.60E-07	3.18E-06
29	1.447694E-01	3.29E-07	2.27E-06
30	1.447713E-01	1.59E-06	1.10E-05
Benchmark price	1.447697E-01		

The terms used in the table are defined in Table 2.

Table 7: Coupon bond call option in the FV (1991) model

Order	Price	Absolute error	Relative error
20	7.229855E-02	3.42E-04	4.70E-03
21	7.242059E-02	2.20E-04	3.02E-03
22	7.251866E-02	1.22E-04	1.67E-03
23	7.258585E-02	5.43E-05	7.48E-04
24	7.262409E-02	1.61E-05	2.21E-04
25	7.263997E-02	2.04E-07	2.80E-06
26	7.264134E-02	1.17E-06	1.60E-05
27	7.263545E-02	4.73E-06	6.51E-05
28	7.262645E-02	1.37E-05	1.89E-04
29	7.261896E-02	2.12E-05	2.92E-04
30	7.261076E-02	2.94E-05	4.05E-04
MC price	7.264017E-02		
Std. Dev.	8.627500E-05		

The terms used in the table are defined in Table 3.

Table 8: Coupon bond call option in the FV (1991) model

Order	Price	Absolute error	Relative error
20	1.101486E-01	3.47E-04	3.16E-03
21	1.099426E-01	1.41E-04	1.29E-03
22	1.098145E-01	1.31E-05	1.19E-04
23	1.097687E-01	3.27E-05	2.98E-04
24	1.097814E-01	2.00E-05	1.82E-04
25	1.098209E-01	1.95E-05	1.78E-04
26	1.098617E-01	6.04E-05	5.50E-04
27	1.098897E-01	8.83E-05	8.04E-04
28	1.099033E-01	1.02E-04	9.28E-04
29	1.099024E-01	1.01E-04	9.20E-04
30	1.099024E-01	1.01E-04	9.20E-04
MC price	1.098014E-01		
Std. Dev.	8.814900E-05		

The terms used in the table are defined in Table 3.

Table 9: Computation time

Method	Computation time (in seconds)
Monte Carlo	692
Approximation	0.753

In the table, *Monte Carlo* refers to a Monte Carlo simulation based on 100,000 sample paths, and *Approximation* is our semi-closed-form solution. Both methods are for the FV (1991) stochastic volatility model.

Figure 1: The weekly 3-month T-bill rates (in percentages) are taken from the H.15 release at the Federal Reserve Board over the sample period of January 1954 to December 2006.

Figure 2: The first order differences of the 3-month T-bill rates plotted in Figure 1.

Figure 3: This graph plots an upward-sloping term structure of interest rates generated by the FV (1991) interest rate model with stochastic volatility. In the figure *Yield* is in decimals and *Maturity* is in years.

Figure 4: This graph draws a downward-sloping yield curve resulted from the FV (1991) stochastic volatility model. In the plot *Yield* is in decimals and *Maturity* is in years.

Figure 5: This plot graphs a humped-shaped term structure of interest rates generated by the FV (1991) interest rate model with stochastic volatility. In the graph *Yield* is in decimals and *Maturity* is in years.

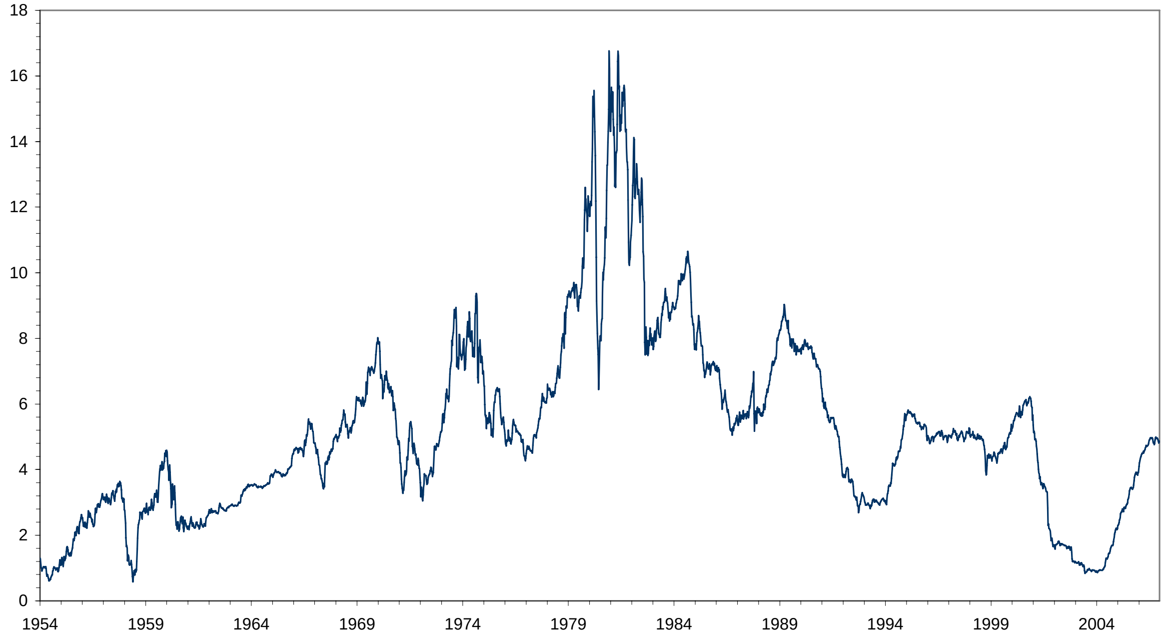

Figure 1: Weekly 3-month U.S. T-bill rates

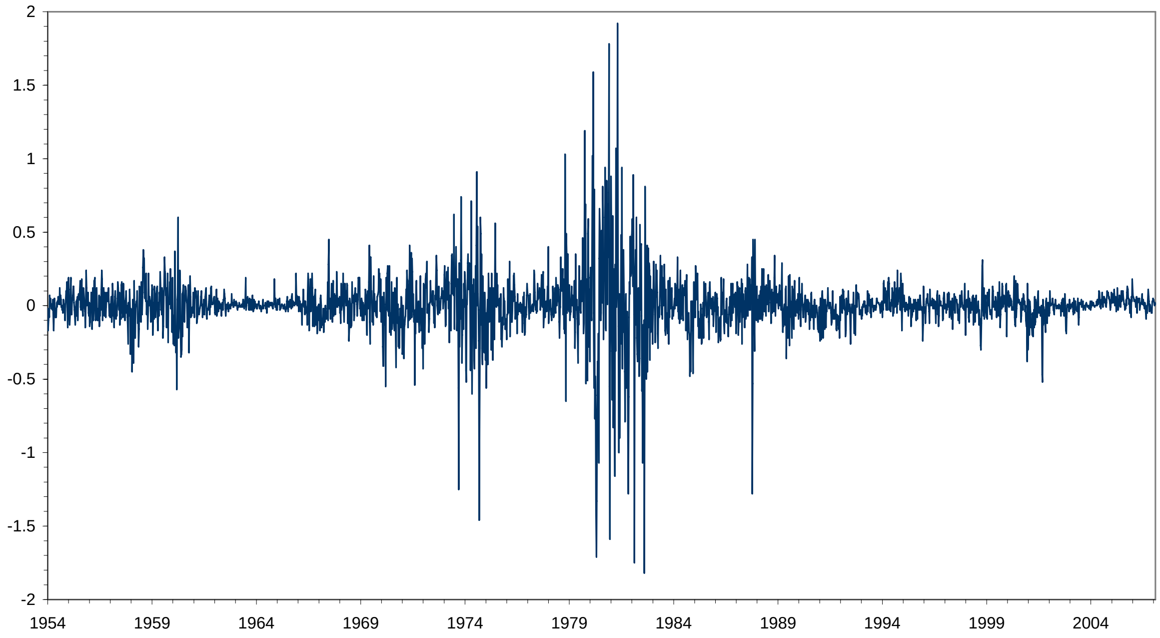

Figure 2: First order changes of weekly 3-month U.S. T-bill rates

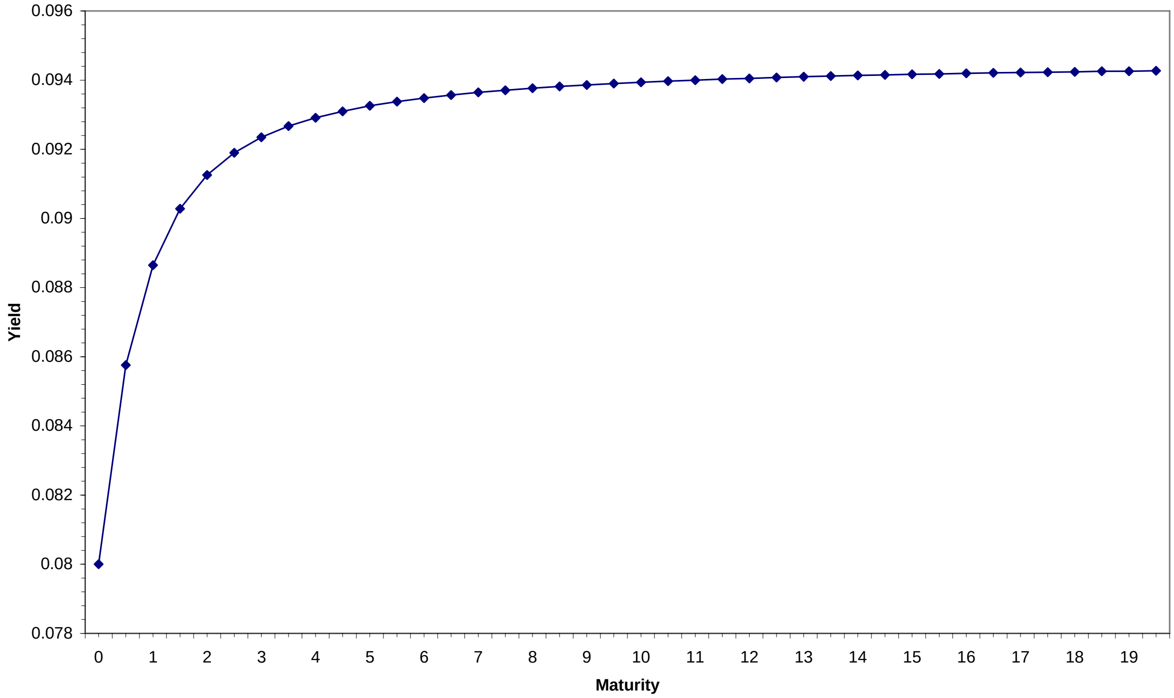

Figure 3: Term structure of interest rates – normal curve

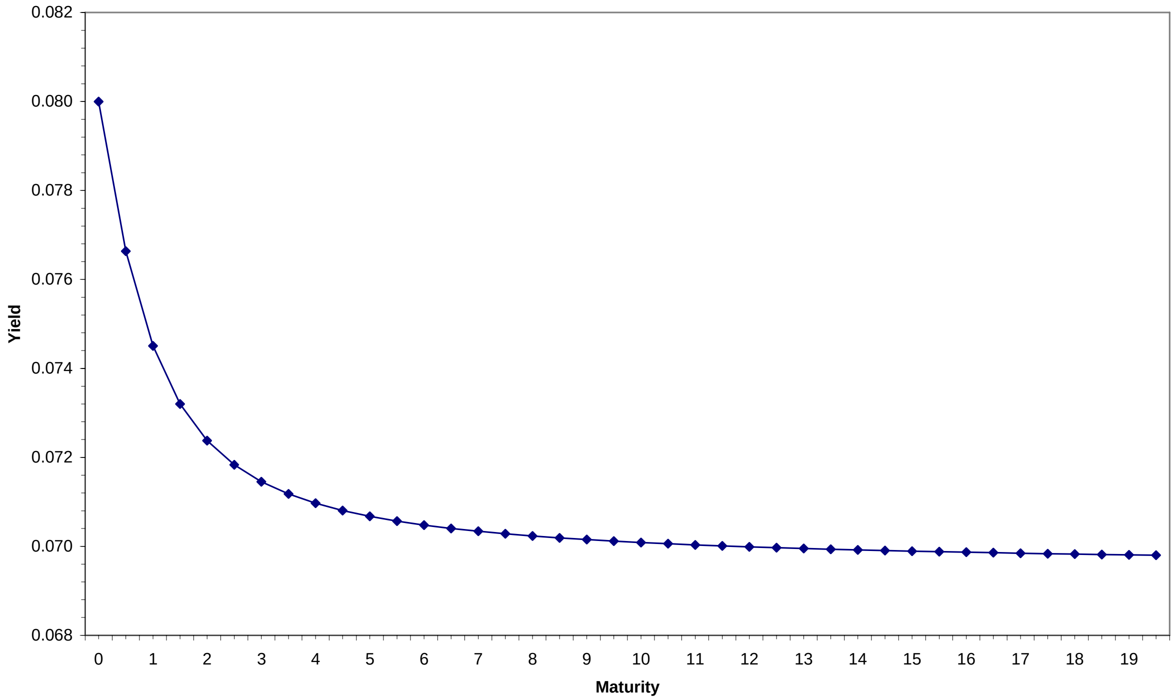

Figure 4: Term structure of interest rates – inverted curve

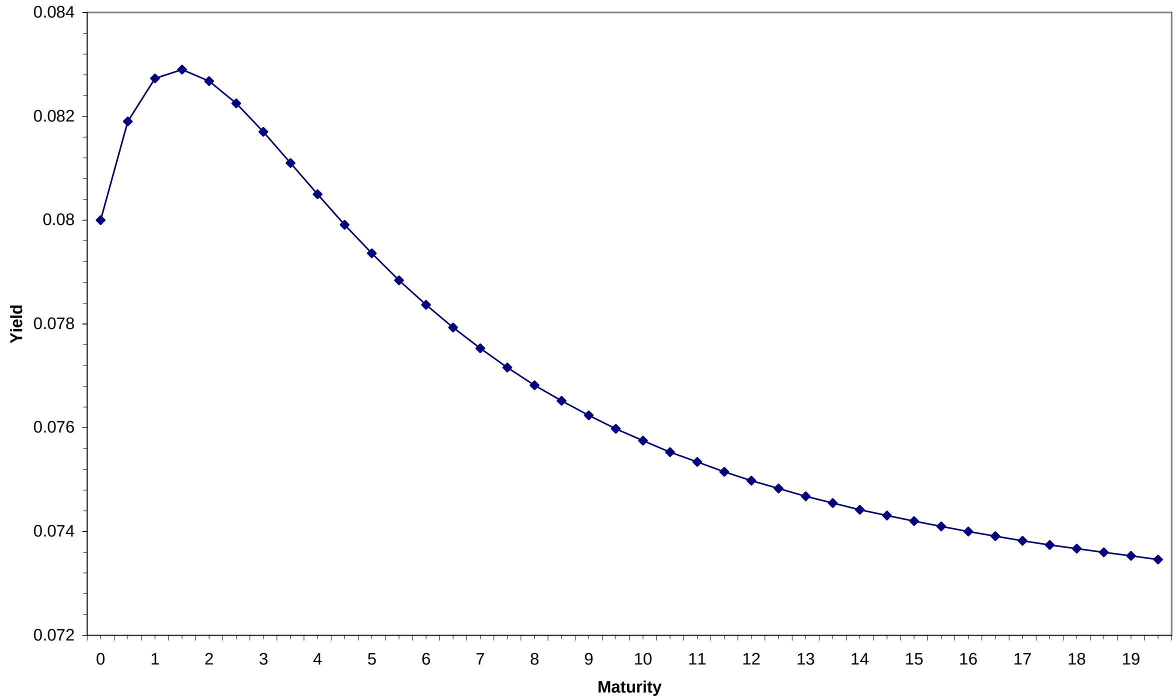

Figure 5: Term structure of interest rates – hump-shaped curve